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Derivation of Both Klein-Gordon Equation and Time Dependent Schrödinger Equation From Classical Wave Function

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Abstract

In this study, both of Klein-Gordon equation and time dependent Schrödinger equation are directly derived from De Broglie hypothesis (λ = h/p). Besides that, a new formula of Klein-Gordon equation was obtained by using two equations, i.e., the classical wave function and relativistic total energy relation hw = (h^2k^2)/2m + V°. Moreover, a new equation of time dependent Schrödinger (TDSE) in three-dimension 3D was derived and evidently obtained. Thus, previous result is corresponding to (TDSE) in 3D calculated by using two equations, (i.e., Einstein energy and Newton second law of mechanics).

Key words-De Broglie hypothesis; relativistic; potential energy; kinetic energy; rest mass; dependent; Special Relativity; general relativity.

I. INTRODUCTION

Einstein theory of special relativity (SR) is one of the biggest achievements in modern physics theory. It changes radically both of the classical concepts in space and time [1], and the generalization made in Einstein generalized special relativity (EGSR) model [2]. Schrödinger approach was yield and abstract the picture of the micro world [3]. Also, this approach could, derive a periodic table of element. It was slightly different form, but close to Mendelevy periodic law, thus based on spherical solution of his standing wave equation. Schrödinger [4], [5], presented a paper in the time dependent phenomena in quantum mechanics conference. The paper confirmed the time dependent phenomena in quantum mechanics. Later, Schrödinger was derived the time dependent Schrödinger equation (TDSE). Which it was discussed the time and space operators in quantum mechanics.

II. DERIVATION OF KLEIN-GORDON EQUATION

The expression of Klein-Gordon equation from relativistic energy and momentum by using Newton second law and wave function showing that in relativistic Einstein energy:

\[ E = mc^2 = \frac{m_e c^2}{\sqrt{1 - \beta^2}} \]

III. RELATION BETWEEN FORCE AND ENERGY

Form Newton second law of mechanics, the force (F) is comprising a definition of relativistic momentum is given by:

\[ dE = F dx \rightarrow E = F \int dx \] (1)

The kinetic energy of particle that is traveling with speed close to the light speed, this particle can make an acceleration beside the momentum according to the equation below:

\[ F = \frac{dp}{dt} = \frac{dmv}{dt} \int \frac{dm}{dt} c \] (2)

Let us v = dx/dt and substituting equation 2 in equation 1 we get:
The Einstein mass and energy is related to this equation. Now, rewrite the equation \(5\) again

\[
E = 
\begin{align*}
&\int \left[\frac{dp}{dt}\right] dx = \int \left[\frac{dmv}{dt}\right] dx \\
= &\int d(mv) \frac{dx}{dt} = \int vdmv \\
E = &\int [vdmmv] - \int [vdmmv]
\end{align*}
\]  

Using this equation:

\[
m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}  \\
= mv^2 - m_o \int_0^v \frac{vdv}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Let us

\[
cos \theta = \frac{v}{c} = \beta \\
dsin \theta = \cos \theta d \theta 
\]

\[
\gamma = \frac{1}{\sqrt{1 - \cos^2 \theta}}  \\
\gamma = \frac{v}{c} = \cos \theta \rightarrow v = c \cos \theta \rightarrow v^2 = c^2 \cos^2 \theta
\]

\[
m = m_o \sqrt{1 - \frac{v^2}{c^2}}  \\
= m_o \int_0^v \frac{vdv}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Insert equation 3 M. Tagabo, M. Dirar[1].

\[
E = mv^2 - m_o \int \frac{ccos \theta \sin \theta d \theta}{\sqrt{1 - \cos^2 \theta}}  \\
E = mv^2 - m_o \int \frac{ccos \theta \sin \theta d \theta}{\sin \theta}  \\
E = mv^2 - m_o c^2 (- \sin \theta)  \\
E = mv^2 - m_o c^2 \frac{1}{\sqrt{1 - \cos^2 \theta}}  \\
E = mv^2 - m_o c^2 \sqrt{1 - \frac{1}{\beta^2}}  \\
E = mv^2 - m_o c^2 \frac{1 - \beta^2}{\sqrt{1 - \beta^2}}  \\
E = \frac{m_o c^2}{\sqrt{1 - \beta^2}} = \gamma m_o v^2 = mc^2 = E
\]

Substituting equation 5 in equation 4

\[
E = mv^2 + mc^2 - mv^2  \\
E = mv^2
\]  

IV. TOTAL ENERGY EQUATION IN RELATIVISTIC

The energy and momentum in SR are related according to the relation in equation 7 Multiplying both sides in equation 7 by \(\psi\) yields

\[
E^2 \psi = c^2 p^2 \psi + m_o c^2 \psi
\]  

The relation between energy-momentum in SR showing in wave function:

\[
\psi = Ae^{\frac{i}{\hbar} (px - Et)}
\]  

Using equation 9 and differentiating both sides with respect to time and space we get:

\[
\frac{\partial \Psi}{\partial x} = \left[ \frac{i}{\hbar} \right] [p] Ae^{\frac{i}{\hbar} (px - Et)}
\]

\[
\frac{\partial^2 \Psi}{\partial x^2} = - \left[ \frac{1}{\hbar^2} \right] [p^2] Ae^{\frac{i}{\hbar} (px - Et)} \rightarrow \frac{\partial^2 \Psi}{\partial x^2} = - \frac{p^2}{\hbar^2} \psi
\]

\[
p^2 \psi = \hbar^2 \frac{\partial^2 \Psi}{\partial x^2}
\]

\[
\frac{\partial \Psi}{\partial x} = - \left[ \frac{i}{\hbar} \right] [E] Ae^{\frac{i}{\hbar} (px - Et)}
\]

\[
\frac{\partial^2 \Psi}{\partial t^2} = - \left[ \frac{1}{\hbar^2} \right] [E^2] Ae^{\frac{i}{\hbar} (px - Et)} \rightarrow \frac{\partial^2 \Psi}{\partial t^2} = - \frac{E^2}{\hbar^2} \psi
\]

\[E \Psi = - \hbar^2 \frac{\partial^2 \Psi}{\partial t^2}
\]  

\[
E^2 \Psi = \left[ -\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} \right]
\]
\[ \left[ -\hbar^2 \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x} \right] \Psi(r, t) = \left[ -\hbar^2 \frac{\partial^2}{\partial \lambda^2} - \frac{1}{\lambda^2} \right] \Psi(r, t) - \frac{m^2 e^2}{\hbar^2} \Psi(r, t) \quad (12) \]

V. DERIVATION OF TIME DEPENDENT SCHRÖDINGER EQUATION

De Broglie suggested a particle with a momentum has an associated wavelength form this relation:

\[ \lambda = \frac{\hbar}{p} \rightarrow p = \frac{h\nu}{c} = \frac{h}{\lambda} \quad (13) \]

And the relation between wave number and wave length

\[ h = 2\pi \hbar, k = \frac{2\pi}{\lambda} \quad (14) \]

Where: \( k \) is wave number of Substituted equation 14 by equation 15, it get:

\[ P = \hbar k \rightarrow p^2 = \hbar^2 k^2 \quad \text{......(h)} \]

VI. RELATION BETWEEN PHOTON ENERGY AND FREQUENCY

According to (GSRE) a traveling plane wave with wave number and angular frequency and photon energy is related form this relation:

\[ E = h\nu, \quad \omega = 2\pi \nu, \quad h = 2\pi \hbar \rightarrow E = [2\pi h] \left[ \frac{w}{2\pi} \right] \quad \text{......(2h)} \]

The kinetic energy \( (E_k) \) plus potential energy \( (E_p) \) equals total energy \( (E) \). \( E = E_k + E_p \)

Now, \( \frac{1}{2} mv^2 \) is the expression for kinetic energy in Newtonian mechanics:

\[ E = \left[ \frac{1}{2} mv^2 + V_o \right] \rightarrow \left[ \frac{1}{2} mv^2 \right] \frac{m}{m} + V_o \quad (15) \]

\[ E = \left[ \frac{m^2 \nu^2}{2m} \right] + V_o = \left[ \frac{p^2}{2m} \right] + V_o \]

Substitute equation (2h) in equation 15 we get:

\[ \hbar w = \left[ \frac{\hbar^2 k^2}{2m} \right] + V_o \quad \text{......(3h)} \]

Since the classical wave equation is given by:

\[ \Psi(x, t) = [\cos(kx - \nu t) + \gamma \sin(kx - \nu t)] \quad (16) \]

\[ \alpha \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + V_o \Psi(x, t) = \beta \left[ \frac{\partial}{\partial t} \right] \Psi(x, t) \quad (17) \]

Multiplying equation 16 with equation 17 to get:

\[ \alpha \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + V_o \Psi(x, t) = \beta \left[ \frac{\partial}{\partial t} \right] \Psi(x, t) \quad (18) \]

\[ \alpha \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + V_o \Psi(x, t) = \beta \left[ \frac{\partial}{\partial t} \right] \Psi(x, t) \quad (19) \]

Rewrite equation 18 again

\[ \alpha \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + V_o \Psi(x, t) = \beta \left[ \frac{\partial}{\partial t} \right] \Psi(x, t) \quad (20) \]

and

\[ \alpha \gamma k^2 \sin(kx - \nu t) - \gamma w \beta \sin(kx - \nu t) = 0 \]

\[ \alpha \gamma k^2 + \gamma w \beta = 0 \quad (21) \]

Rewrite equation 20 & 21 again and left side in equation 21 equal left side in equation 21:

\[ -\alpha k^2 + V_o = -\gamma w \beta \quad \text{......(4h)} \]

\[ \alpha \gamma k^2 + \gamma w \beta \quad \text{......(5h)} \]

\[ \alpha \gamma k^2 = \frac{\hbar^2 k^2}{2m} \rightarrow \alpha = \frac{\hbar^2}{2m} \quad (23) \]

Comparing left side and substituting quantity of \( \gamma \) form equation 22 we get:

\[ -i\omega \beta = \hbar w \rightarrow \beta = \frac{h}{\omega} \text{ and } i^2 = -1 \]

\[ \beta = \left[ \frac{1}{i} \right] h \rightarrow \left[ \frac{2\pi}{i} \right] \hbar \rightarrow i\hbar \]

Substitute equation 22 and equation 23 and equation 24 in equation 17.

\[ \alpha = \frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + V_o \Psi(x, t) = \beta \left[ \frac{\partial}{\partial t} \right] \Psi(x, t) \quad (24) \]
In equation 25, the time evolution of $\Psi(x, t)$ is described by the time dependent Schrödinger equation in one dimension, now rewrite equation 25 in three dimensions (3D) to become 26.

$$\left[\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right] \Psi(x, t) + [V_o] \Psi(x, t) - \left[ i\hbar \frac{\partial}{\partial t} \right] \Psi(x, t) \tag{25}$$

**VII. CLASSICAL WAVE FUNCTION**

This is the second method used to express time dependent Schrödinger equation from classical wave function. Several ways can be used to express and interpret the time dependent Schrödinger equation. It is different method but the result is similar. One of this method is the classical wave function, which is related to this equation:

$$\frac{\partial^2}{\partial x^2} \Psi(x, t) = \left[ \frac{1}{v_{ph}^2} \right] \left[ \frac{\partial^2}{\partial t^2} \right] \Psi(x, t) \tag{26}$$

Where $v_{ph} = \text{phase speed}$ The classical wave function can describe time equation, by used below equation: $F(t) = \cos 2\pi vt$ The relation between $(v)$ and $v_{ph}$ is given by wave length $\lambda = \frac{1}{\nu_{ph}} \ldots (6\hbar)$

Now, we can multiply both side of equation 26 by $F(t)$, and substituting left side in equation 27 with equation 26:

$$(t) \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + \left[ \frac{1}{v_{ph}^2} \right] \left[ \frac{\partial^2}{\partial t^2} \right] F(t) \Psi(x, t) \tag{27}$$

$F(t) \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) = \left[ \frac{1}{v_{ph}^2} \right] \left[ \frac{\partial^2}{\partial t^2} \right] \cos 2\pi vt \tag{28}$

$F(t) \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) + \left[ \frac{1}{v_{ph}^2} \right] \left[ \frac{\partial^2}{\partial t^2} \right] [2\pi \nu \sin 2\pi vt] \Psi(x, t) \tag{29}$

$F(t) \left[ \frac{\partial^2}{\partial x^2} \right] \Psi(x, t) = \left[ \frac{1}{v_{ph}^2} \right] \left[ \frac{\partial^2}{\partial t^2} \right] [4\pi^2 \nu^2 \cos 2\pi vt] \Psi(x, t) \tag{30}$

Rewrite this equation again and dividing both side by $F(t)$

$$\frac{\partial^2}{\partial x^2} = - \left[ \frac{v^2}{v_{ph}^2} \right] [4\pi^2] \tag{31}$$

Substitute equation (6h) in equation 28

$$\frac{\partial^2}{\partial x^2} = \left[ \frac{1}{\lambda^2} \right] [4\pi^2] \rightarrow - \left[ \frac{4\pi^2}{\lambda^2} \right] \rightarrow - \left[ \frac{2\pi^2}{\lambda} \right]^2 \tag{29}$$

$$\frac{\partial^2}{\partial x^2} = - \left[ \frac{2\pi p}{\hbar} \right]^2 \rightarrow - \left[ \frac{2\pi p}{2\pi h} \right]^2 \rightarrow \frac{p^2}{\hbar^2} \tag{30}$$

$$p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \tag{31}$$

Substitute equation 13 in this equation. Substitute equation 11 and equation 29 in equation 17

$$\left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} \right) \right] \Psi(x, t) = [V_o] \Psi(x, t) - \left[ i\hbar \frac{\partial}{\partial t} \right] \Psi(x, t) \tag{30}$$

$= \left[ \frac{\hbar^2}{2m} \right] \Psi(x, t) = [V_o] \Psi(x, t) - \left[ i\hbar \frac{\partial^2}{\partial x^2} \right] \Psi(x, t)$

The first expression in this paper showing in equation 11 expression of Klein-Gordon equation by using Newton second law

$$\left[ \frac{1}{\gamma^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m_e^2 c^2}{\hbar^2} J \right] \Psi(r, t) = 0$$

The second expression of time dependent Schrödinger equation showing in equation 25 by using this

$$\alpha \left[ \frac{\partial^2}{\partial t^2} \Psi(x, t) + V_o \Psi(x, t) = \beta \frac{\partial}{\partial t} \Psi(r, t) \right]$$

Obtained new formula of time dependent Schrödinger equation

$$\left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} \right) \right] \Psi(x, t) = [V_o] \Psi(x, t) - \left[ i\hbar \frac{\partial}{\partial t} \right] \Psi(x, t) \tag{31}$$

The third expression of time dependent Schrödinger equation showing in equation 30 by using this relation

$$\frac{\partial^2}{\partial x^2} \Psi(x, t) = \left[ \frac{1}{v_{ph}^2} \right] \frac{\partial^2}{\partial x^2} \Psi(x, t) \tag{32}$$

obtained new formula of time dependent Schrödinger equation

$$\left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} \right) \right] \Psi(x, t) = [V_o] \Psi(x, t) - \left[ i\hbar \frac{\partial}{\partial t} \right] \Psi(x, t) \tag{33}$$

**VIII. DISCUSSION**

The Klein-Gordon equation is directly derived from many formula of Newton second law of mechanics and relativistic total energy relation and relativistic Einstein

$$E = mc^2 = \left[ \frac{m_e c^2}{\sqrt{1 - \beta^2}} \right]$$

Equations 6 & 7 also express Klein-Gordon equation form wave function showing in equation 12 in other side expression of time dependent Schrödinger equation by tow way the first method dividing this

$$\frac{\partial^2}{\partial x^2} \Psi(x, t) + V_o \Psi(x, t) = \beta \frac{\partial}{\partial x} \Psi(x, t)$$

To obtained time dependent Schrödinger (TDSE) in 3D showing in equation 25. And second method using this formula

$$\frac{\partial^2}{\partial x^2} \Psi(x, t) = \left[ \frac{1}{v_{ph}^2} \right] \frac{\partial^2}{\partial x^2} \Psi(x, t)$$

To obtained time dependent Schrödinger (TDSE) in 3D showing in equation 30

**IX. CONCLUSION**

In this paper we are expression and derivation of both Klein-Gordon equation and time dependent Schrödinger equation. We used a different method in this study, such as new definition of force, classical wave function, beside Lorentz vector for $\gamma = i = 1 = \sqrt{1-}$ all of these are obtained similar and compatible three equation. One of them for Klein-Gordon equation, and other two equations for time dependent Schrödinger equation (TDSE) in 3D.
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