

Expression Time Dependent Schrödinger Equation from De Broglie Hypothesis and Hamiltonian Function

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Abstract

The Schrödinger and de Broglie hypothesis are directly derived from classical force and blank hypothesis. Thus, depending on the quantum equations the quantum of total energy relation ($E = T + V$) and de Broglie relation ($\lambda = h/p$). Accordingly, new time dependent Schrödinger equation is obtained. As a result, by using the wave function and nonrelativistic wave function beside plank hypothesis. Moreover, TDSE was obtained, thus by used additional kinetic energy and potential energy (classical equation). Finally, a result corresponding to TDSE in 3D was confirmed by using De Broglie hypothesis.

Key words-Time dependent, classical wave function, relativistic, hypothesis, particle, quantization.

I. INTRODUCTION

For non-relativistic quantum physics the basic equation to be solved is the Schrödinger equation [1]. Like Newton laws the Schrödinger equation must be written down for a given situation of a quantum article moving under the influence of some external force. Quantum mechanics is an essential part of undergraduate syllabus in physics as well as in chemistry [2]. The time dependent phenomena in quantum mechanics was presented in conference on the time dependent Schrödinger equation (TDSE). The time in the equation is not an operator in quantum mechanics, but space is an operator and therefore, the TDSE as:

$$i\hbar \frac{\partial \Psi(r,t)}{\partial t} = H(\vec{r}, t) \Psi(\vec{r}, t)$$

The previous equation is strange mixture of classical (C number) time and quantum (operator) space [3]. Schrödinger approach yields abstract phenomenological constructions which do not reflect the real picture of the micro world [4],[5]. Derive a periodic table of element (slightly different form but close to Mendeleyev periodic law) based on spherical solution of his standing wave equation [6],[7]. In this research relativistic quantum mechanics law will review and explain. Then, time dependent Schrödinger equation will discuss by using many formulae of wave function and nonrelativistic classical wave function beside plank relation to obtained new wave equation of TDSE in 3D.

II. ABBREVIATION

- 1) TDSE time dependent Schrödinger equation.
- 2) DBHB de Broglie hypothesis.
- 3) 1-D, 3-D one dimensional and three dimensional.
- 4) E_t, E_k, E_p total energy and kinetic energy and potential energy.
- 5) m_o, E_o rest mass and rest energy.

III. MATERIAL AND METHOD

Used this section to drive time de pendent Schrödinger equation Abbreviate TDSE. Firstly, used the explanation of de Broglie relation form plank hypothesis to find wave function. Then, using wave function and de Broglie hypothesis to achieve TDSE expression in three dimension (3D).

IV. EXPLANATION OF DE BROGLIE HYPOTHESIS BY USING PLANK HYPOTHESIS

Since the electromagnetic wave can be assumed as a photon moving with the speed of light C, the photon energy is given by:

$$E = \hbar \omega = m_o c^2 = \hbar \omega \quad (1)$$

$$E = \frac{1}{2} m v^2 + V(x) \quad (2)$$

Substituting equation (Eq.1) in equation (Eq.2) and differentiating by velocity v if the body in rest mass $V = 0$

$$\hbar \omega = \frac{1}{2} m v^2 + V(x) \hbar \frac{dw}{dv} = m v \quad (3)$$

Then, the photon momentum is given by:

$$\begin{aligned}
 p &= mv = \hbar \frac{dw}{dv} \frac{dk}{dv} \quad (4) \\
 \frac{dw}{dk} &= v \rightarrow \frac{dw}{dv} \cdot \frac{dv}{dk} = v = k \frac{dv}{dk} \\
 \frac{\hbar k}{m} &= k \frac{dv}{dk} \rightarrow \frac{\hbar}{m} = \frac{dv}{dk} \\
 \frac{\hbar}{m} &= \frac{dv}{dk} \rightarrow \frac{m}{\hbar} = \frac{dk}{dv} \\
 k \frac{mv}{\hbar} &= \frac{P}{\hbar} = \frac{2\pi}{\lambda} \rightarrow \frac{2\pi}{\lambda} = \frac{P}{\hbar} \\
 \hbar &= 2\pi \hbar \\
 E &= \hbar w \\
 P &= \hbar k \\
 w &= 2\pi v \\
 k &= \frac{2\pi}{\lambda} \\
 \lambda &= \frac{c}{v}
 \end{aligned}$$

Where w is the angular frequency and K is the wave vector. The magnitude of $K = k$ equal to the wave number of energy and momentum for this equation. From the De- Broglie equation, the relation between energy and momentum of matter waves: $(\hbar w)^2 = (c\hbar k)^2 + (m_o c^2)^2$ Dividing by:

$$\left(\frac{w}{c}\right)^2 = k^2 + \left(\frac{m_o c^2}{\hbar}\right)^2$$

Since the reduced Planck constant $\hbar$ and speed of light C . Both appear and clutter in equation are natural units, so the normalizing is can be write as $\hbar = c = 1$ So, the equation can be write as:

$$\begin{aligned}
 w^2 &= k^2 + m_o^2 \rightarrow w^2 = c^2 k^2 + \frac{m_o c^4}{\hbar^2} \quad (5) \\
 E^2 &= c^2 p^2 + m_o c^2 \\
 \lambda &= \frac{2\pi \hbar}{p} \rightarrow \lambda = \frac{\hbar}{p}
 \end{aligned}$$

V. EXPLANATION OF WAVE FUNCTION

The wave function associated with particle moving a long (+x) direction is given as [1].

$$y = A \sin \frac{2\pi v}{\lambda} \left(t - \frac{x}{v}\right) \quad (6)$$

$$w = 2\pi v, v = v\lambda$$

$$\Psi = A e^{-i w(t - \frac{x}{v})}$$

$$\Psi = A e^{-i 2\pi w(t - \frac{x}{v\lambda})} = A e^{-i 2\pi v(vt - \frac{x}{\lambda})} \quad (7)$$

Form equation (1) the energy is given by:

$$E = \hbar w = 2\pi \hbar v \rightarrow \hbar = \frac{\hbar}{2\pi} \rightarrow v = \frac{E}{2\pi \hbar}$$

Where h is plank constant Also, equation (Eq.5) equal to:

$$\frac{2\pi \hbar}{p} \rightarrow \lambda = \frac{h}{P}$$

$$\Psi = A e^{-i 2\pi \left(\frac{Et}{2\pi \hbar} - \frac{XP}{2\pi \hbar}\right)}$$

$$\Psi = A e^{-\frac{i 2\pi}{2\pi \hbar} (Et - XP)}$$

$$\Psi = A e^{-\frac{i}{\hbar} (Et - XP)} \rightarrow A e^{\frac{i}{\hbar} (PX - Et)} \quad (8)$$

Interpretation equation (Eq.5) with wave particle duality:

a. Particle proprieties (electron) the particle energy is given by:

$$E_k = \frac{1}{2} m v^2 = \frac{m \cdot m v^2}{2m} = \frac{m^2 v^2}{2m} = \frac{p^2}{2m} \quad (9a)$$

If the particle velocity $v < c$ than the square of energy, then the total energy is: $E^k = \frac{1}{2} m v^2 + m_*^2 c^2$ Assumed the particle in rest $v=0$ then:

$$E^2 = m_*^2 c^2 E^* = \frac{1}{2} m v^2 + V(x) \quad (10)$$

b. Wave particles $\lambda = \frac{h}{p} \rightarrow E = hf$ If the wave moving by $c = \lambda f$ Then:

$$c = \lambda f = \left[\frac{h}{mv}\right] \left[\frac{1}{2} \frac{mv^2}{h}\right] = \frac{v}{2} \quad (11)$$

The equation implies the wave velocity moving by half to velocity of particle.

VI. DERIVATION TDSE FROM DE BROGLIE HYPOTHESIS

The time dependent Schrödinger equation (TDSE) can be derived from the de Broglie hypothesis. Also, by used plank relation. Then, the general equation can be used to describes particles vibration in tight string and that is not depend to time t . By add some modifications to Drar, Tagabo [5] equation. New formula can be written as:

$$\frac{1}{\lambda} \sqrt{1 + \frac{D}{2}} = -\frac{d^2 \vartheta}{4\pi^2 \vartheta} \quad (12)$$

Let us

$$\begin{aligned}
 x &= \frac{1}{\lambda} \sqrt{1 + \frac{D}{2}} \\
 x^2 &= \frac{1}{\lambda^2} \left[1 + \frac{D}{2}\right] \rightarrow dx^2 = \frac{1}{\lambda^2} d \left[1 + \frac{D}{2}\right]
 \end{aligned}$$

Substitute in equation (Eq.??)

$$\frac{1}{\lambda^2} dx^2 = -\frac{d^2 \vartheta}{4\pi^2 \vartheta} \dots (11h)$$

Where ϑ is non relativistic classical wave function now arrangement equation (Eq.??) again.

$$\frac{d^2 \vartheta}{dx^2} = -\frac{4\pi^2 \vartheta}{\lambda^2} \quad (13)$$

Substitute equation (Eq.5) in equation (Eq.13)

$$\frac{d^2 \vartheta}{dx^2} = -\frac{4\pi^2 p^2 \vartheta}{\hbar^2} \quad (14)$$

Equation (Eq.3) is classical equation in the one dimensional can be written in three dimensional

$$\begin{aligned}
 \left[\frac{d^2 \vartheta}{dx^2} + \frac{d^2 \vartheta}{dy^2} + \frac{d^2 \vartheta}{dz^2}\right] &= -\frac{4\pi^2 p^2 \vartheta}{\hbar^2} \\
 \Delta^2 \vartheta &= -\frac{4\pi^2 p^2 \vartheta}{\hbar^2} \quad (15)
 \end{aligned}$$

Now using equation (Eq.1a) the total energy equals the kinetic energy plus potential energy

$$E_t = E_k + E_p \quad (16)$$

The kinetic energy in classical force is equal equation (Eq.1a)

$$E_k = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

Hence equation (Eq.1) Rewrite again and become:

$$E_k = E_t - E_p \quad (17)$$

Left side in equation (Eq.1a) equal left side in equation Eq.17) can be written

$$\frac{p^2}{2m} = [E_t - E_p] \rightarrow p^2 = 2m [E_t - E_p]$$

Substituting P2 in equation (Eq.15)

$$\begin{aligned} \frac{d^2\vartheta}{dx^2} &= -\frac{[(4\pi^2)]2m[E_t - E_p]\vartheta}{h^2} \rightarrow h = 2\pi\hbar \\ &\rightarrow \frac{[(4\pi^2)]}{h^2} = \frac{1}{\hbar^2} \\ \frac{d^2\vartheta}{dx^2} &= -\frac{2m}{\hbar^2} [(E_t - E_p)]\vartheta \\ -\frac{\hbar^2}{2m} \frac{d^2\vartheta}{dx^2} &= [E_t - E_p]\vartheta \end{aligned} \quad (18)$$

Using equation (Eq.8) and replace classical wave function by wave function

$$\Psi = Ae^{\frac{i}{\hbar}(PX-ET)}$$

Differentiation equation (Eq.8) with respect to time

$$\frac{\partial\Psi}{\partial t} = -E\frac{i}{\hbar}PAe^{\frac{i}{\hbar}(PX-ET)} = \frac{iE}{\hbar}\Psi \rightarrow E\Psi = -\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} \quad (19)$$

$$E\Psi = (Et - Ep)\Psi - \frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} = -\frac{\hbar}{i} \frac{\partial\Psi}{\partial t}$$

Equation (Eq.10) become:

$$-\frac{\hbar}{i} \frac{\partial\Psi}{\partial t} = -\left[\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2}\right] + V(x)$$

V(x) is potential energy of particle in one dimensional:

$$-\frac{\hbar}{i} \frac{\partial\Psi}{\partial t} = \left[\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2}\right] - V(x) \quad (20)$$

This approach of deriving TDSE form de Broglie hypothesis in one dimensional (1 D) can be written in three dimensional (3 D)

$$-\frac{\hbar}{i} \frac{\partial\Psi}{\partial t} = \frac{\hbar^2}{2m} \left[\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dY^2} + \frac{d^2\Psi}{dZ^2}\right] - V(x) \dots\dots\dots (2a)$$

VII. USING WAVE NUMBER TO DERIVED TDSE:

If the electron particle moving by mass m and potential energy V(x) the classical potential energy become: $V = -\frac{kq^2}{r}$

Now replace the charge q by electron charge e we get:

$$V = -\frac{ke^2}{r} \rightarrow k\frac{1}{4\pi\epsilon_0} \rightarrow V = -\frac{e^2}{r4\pi\epsilon_0}$$

Equation (2) become:

$$E = \frac{1}{2}mv^2 - \frac{e^2}{r4\pi\epsilon_0} E = \frac{1}{2} \left[\frac{p^2}{m}\right] - \left[\frac{e^2}{r4\pi\epsilon_0}\right] \quad (21)$$

For free particles where V(x)=0 the wave function solution can be put in the form plank wave:

$$\Psi(x, t) = A(e^{ax}.e^{bt})$$

And equation (20) become: $E\Psi = \frac{p^2}{2m}\Psi$ Proceeding separately for the position and time t equation and taking the indicated derivatives (a)

$$\begin{aligned} -\frac{\hbar^2}{2m}a^2\Psi &= EM \\ a^2 &= -2\frac{EM}{\hbar^2} \rightarrow a = \sqrt{-2\frac{EM}{\hbar^2}} = i\sqrt{2\frac{EM}{\hbar^2}} \end{aligned}$$

For kinetic energy:

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} \rightarrow a = \sqrt{\frac{p^2}{\hbar^2}} = i\frac{p}{\hbar} \quad (22)$$

For de Broglie relationship and wave relationship

$$a = \frac{ih}{\hbar\lambda} = \frac{2i\pi}{\lambda}$$

$$E\Psi = ib\hbar\Psi \rightarrow b = -\frac{iE}{\hbar}$$

For plank hypothesis gives $E = \hbar\omega$ we can evaluate the constant $b = -i\omega$ the general free particle wave function:

$$\Psi(x, t) = Ae^{\lambda i2\pi - i\omega t} = Ae^{ikx - i\omega t} \quad (23)$$

Now multiplying equation (Eq.??) by equation (Eq.3a) and replace p to $\hbar k$

$$\begin{aligned} [E]Ae^{ikx - i\omega t} &= \left[\frac{\hbar^2 k^2}{2m}\right] Ae^{ikx - i\omega t} \\ &- \left[\frac{e^2}{r4\pi\epsilon_0}\right] Ae^{ikx - i\omega t} \end{aligned} \quad (24)$$

Differentiating equation (Eq.13) with t and with x

$$\begin{aligned} \frac{\partial\Psi}{\partial t} &= i\omega\Psi \rightarrow \omega = -\frac{1}{i} \frac{\partial\Psi}{\partial t} \\ \frac{\partial\Psi}{\partial x} &= ik\Psi \rightarrow \frac{\partial^2\Psi}{\partial x^2} = -K^2\Psi \rightarrow K^2\Psi = -\frac{\partial^2\Psi}{\partial x^2} \end{aligned} \quad (25)$$

Rewrite equation (Eq.13):

$$\begin{aligned} -\frac{\hbar}{i} \frac{\partial\Psi}{\partial t} &= \frac{\hbar^2}{2m} \frac{\partial^2\Psi}{\partial x^2} - \frac{e^2}{r4\pi\epsilon_0}\Psi \rightarrow V(x) = -\frac{e^2}{r4\pi\epsilon_0} \\ &-\frac{\hbar}{i} \frac{\partial\Psi}{\partial t} = -\left[\frac{\hbar^2}{2m} \frac{\partial^2\Psi}{\partial x^2}\right] + V(x) \\ \frac{\hbar}{i} \frac{\partial\Psi}{\partial t} &= \left[\frac{\hbar^2}{2m} \frac{\partial^2\Psi}{\partial x^2}\right] - V(x)\Psi(x, t) \end{aligned} \quad (26)$$

Equation (Eq.??) is the TDSE in (1D) now write in (3D).

$$\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = \frac{\hbar^2}{2m} \left[\frac{\partial^2 \Psi}{\partial x^2} \frac{\partial^2 \Psi}{\partial y^2} \frac{\partial^2 \Psi}{\partial z^2} \right] - V(x) \Psi(x, t) \quad (27)$$

VIII. DISCUSSION

The time dependent Schrödinger equation are directly derived from many formulas of wave function and relativistic wave function. The quantity in expression $(p^2/2m)$, the square of momentum divided by particle mass is obtained through the classical kinetic energy and showing in equation (Eq.1a). Also expression of time dependent Schrödinger equation by using wave function and classical equation (Eq.??) to obtained TDSE in 3D showing in equation (Eq.2a). The equation (Eq.3a) is implying classical wave function equation related between frequency and wave number. Using photon of momentum $p = \hbar k$ to expression of plank hypothesis showing that in equation (Eq.5) to obtained time dependent Schrödinger equation showed in equation (Eq.4a) also, using plank energy of particle in one dimensional in equation (Eq.1). To found relativistic total energy in equation (Eq.17) by this equation and expression TDSE in (1D) that shown in equation (Eq.20). New derived of time dependent Schrödinger equation TDSE in (3D) by using different method but the result as shown in equation (Eq.2a , Eq.4a). The TDSE is constant with energy conservation and with free particle. The potential energy $V(x) = 0$ compared with the Hamilton an equal total energy. Also, the TDSE is constant with electron charge compared with the total energy equals the kinetic energy in this equation: $-\frac{\hbar^2}{2m} \nabla^2 \Psi = E_T \Psi$

IX. CONCLUSION

In this paper interpretation and explanation of time dependent Schrodinger equation from different method .Firstly,the study used plank hypothesis (TDSE) in (3D). After that, derived both of wave function and non-relativistic wave function to confirm (TDSE) in (3D).

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